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LETTER TO THE EDITOR

A quantum integrable supersymmetric multiple three-wave interaction model

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Abstract. A supersymmetric multiple three-wave interaction model is proposed, and its quantum integrability is demonstrated in a straightforward way.

Recently, Ganoulis [1] proposed a quantization prescription for Toda systems using the classical Yang–Baxter equation (CYBE) [2–4]. At present, this prescription has been successfully applied to the study of the so-called Gaudin models [5, 6] associated with the non-degenerate solutions of the CYBE. In particular, Jurco [7] proposed an integrable quantum system which he called 'the quantum multiple three-wave interaction model'.

Taking into account the recent interest in supersymmetry, in this letter we shall propose a quantum integrable supersymmetric multiple three-wave interaction model corresponding to Lie superalgebra $gl(p, q)$, which has boson fields as well as fermion fields.

The system under consideration is described by the Hamiltonian

$$H=\sum_{\alpha=1}^{M}\left[\sum_{i=1}^{p-1}(u_i a_i^{\alpha+}a_p^{\alpha}+u_i a_i^{\alpha}a_p^{\alpha+})+\sum_{j=1}^{q}(a_p^{\alpha+}b_j^{\alpha+}v_j+a_p^{\alpha}b_j^{\alpha}v_j^{+})\right.$$
$$\left.+\tfrac{1}{2}\delta_\alpha\left(a_p^{\alpha+}a_p-\sum_{i=1}^{p-1}a_i^{\alpha+}a_i^{\alpha}-\sum_{j=1}^{q}b_j^{\alpha}b_j^{\alpha+}\right)\right] \tag{1}$$

with δ_α ($\alpha=1,2,\ldots,M$) being mutually distinct real numbers. Here $a_i^{\alpha+}$, a_i^{α}, u_i^{+}, and u_i are boson fields, satisfying the usual commutation relations

$$[a_i^{\alpha+}, a_j^{\alpha}]=-\delta_{ij} \tag{2a}$$

$$[a_i^{\alpha+}, u_j^{+}]=0 \tag{2b}$$

$$[a_i^{\alpha}, u_j]=0 \tag{2c}$$

while $b_j^{\alpha+}$, b_j^{α}, v_j^{+} and v_j are fermion fields, satisfying the anticommuting relations:

$$[b_i^{\alpha+}, b_j^{\alpha}]_{+}=\delta_{ij} \tag{3a}$$

$$[b_i^{\alpha+}, v_j^{+}]=0 \tag{3b}$$

$$[b_i^{\alpha}, v_j]=0. \tag{3c}$$

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The aim of this letter is to show that this system is completely integrable. For convenience, we use the following matrix notation

$$A^\alpha = \begin{pmatrix} X^\alpha & Y^{\alpha+} \\ -Y^\alpha & Z^\alpha \end{pmatrix}. \tag{4}$$

Here X^α is a $P \times P$, Y^α a $q \times p$, Z^α a $q \times q$ matrix:

$$X^\alpha_{ij} = -a_i^{\alpha+} a_j^\alpha \qquad Y^\alpha_{ij} = b_i^\alpha a_j^\alpha \qquad (i, j = 1, 2, \ldots, p)$$

and

$$Z^\alpha_{kj} = b_k^\alpha b_j^{\alpha+} \qquad (k = p+1, \ldots, p+q).$$

Further we introduce

$$U(\lambda) = \lambda U_1 + U_0 \tag{5}$$

where U_1 and U_0 are

$$U_1 = \tfrac{1}{2}\,\mathrm{diag}(\underbrace{-1, \ldots, -1}_{p-1}, 1, \underbrace{-1, \ldots, -1}_{q}) \tag{6}$$

and

$$U_0 = \begin{pmatrix} & U_1^+ & \\ & \vdots & \\ & U_{p-1}^+ & \\ U_1, \ldots, U_{p-1} & 0 & V_1^+, \ldots, V_q^+ \\ & -V_1 & \\ & \vdots & \\ & -V_q & \end{pmatrix} \tag{7}$$

respectively. Thus, we may write out the L matrix for the model (1) in the form

$$L(\lambda) = U(\lambda) + A(\lambda) \tag{8}$$

with

$$A(\lambda) = \sum_{\alpha=1}^{M} \frac{A^\alpha}{\lambda - \delta_\alpha}. \tag{9}$$

After a tedious but straightforward algebraic calculation, we can show that this $L(\lambda)$ matrix satisfies the following CYBE

$$[L(\lambda) \underset{s}{\otimes} I, I \underset{s}{\otimes} L(u)] + [\gamma(\lambda - u), L(\lambda) \underset{s}{\otimes} I + I \underset{s}{\otimes} L(u)] = 0 \tag{10}$$

with

$$\gamma(\lambda - u) = \sum_{l,m=1}^{p+q} (-1)^{p(l)p(m)} \frac{e_{lm} \otimes e_{ml}}{\lambda - u}. \tag{11}$$

Here I is the $(p+q) \times (p+q)$ identity matrix, and e_{lm} the generators of the Lie superalgebra $\mathrm{gl}(p, q)$ in the fundamental representation, $(e_{lm})_{\alpha\beta} = \delta_{l\alpha}\delta_{m\beta}$. Also we note that, the supertensor product '$\otimes_s$' should be understood in the sense of Grassmann [8]:

$$(A \underset{s}{\otimes} B)_{ij,kl} = (-1)^{p(j)[p(i)+p(k)]} A_{ik} B_{jl} \tag{12}$$

with

$$p(1)=p(2)=\ldots=p(p)=0 \tag{13a}$$

$$p(p+1)=p(p+2)=\ldots=p(p+q)=1. \tag{13b}$$

We emphasize that in equation (10) the matrix commutators are taken with respect to the operator nature of matrix elements.

Now we may conclude that the Hamiltonian (1) is related to the $L(\lambda)$ matrix via

$$H=\tfrac{1}{2}\operatorname*{res}_{\lambda=\infty}\operatorname{str}L^2(\lambda)\equiv\tfrac{1}{2}\operatorname*{res}_{\gamma=\infty}\operatorname{str}T(\lambda). \tag{14}$$

In fact, str $T(\lambda)$ may be viewed as the generating functional for conserved quantities:

$$[\operatorname{str}T(\lambda),\operatorname{str}T(u)]=0. \tag{15}$$

Here 'str' denotes the supertrace, defined by

$$\operatorname{str}T(\lambda)=\sum_j(-1)^{p(j)}T_{jj}(\lambda)\qquad j=(1,2,\ldots,p,p+1,\ldots,p+q).$$

Correspondingly, the Heisenberg equations may be cast into the quantum Lax form:

$$\frac{\mathrm{d}}{\mathrm{d}t}L(u)=\mathrm{i}[H,L(u)]=\mathrm{i}[B(u),L(u)] \tag{16}$$

with

$$B(u)=-\operatorname*{res}_{\lambda=\infty}\operatorname{str}_1[\gamma(\lambda-u)L(\lambda)\underset{s}{\otimes}I] \tag{17}$$

where the first commutator in (16) is an operator commutator while the second commutator is a matrix one. The supertrace str in (17) is taken over the first factor in $\mathrm{gl}(p,q)\underset{s}{\otimes}\mathrm{gl}(p,q)$.

So far, we have completed the demonstration of the quantum integrability of the supersymmetric multiple three-wave interaction model proposed in the present letter. Some profound aspects of this model, especially the derivation of the algebraic Bethe ansatz, are now in progress.

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